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CONTINUED FRACTION REPRESENTATIONS OF THE INCOMPLETE GAMMA FUNCTION

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Abstract. In this paper, we summarize the basic definitions and results of matrices, matrix functions, and continued fractions. In the convergent case, the continued fractions expansions have the advantage that they converge more rapidly than other numerical algorithms. Further, this article aims to give a continued fraction expansion of the incomplete gamma function.

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1. Introduction and motivation

Over the last two hundred years, the theory of continued fractions has been a topic of extensive study. The basic idea of this theory over real numbers is to give an approximation of various real numbers by the rational ones. A continued fraction is an expression obtained through an iterative process of representing a number as the sum of its integer part and the reciprocal of another number, then writing this other number as the sum of its integer part and another reciprocal, and so on. One of the main reasons why continued fractions are so useful in computation is that they often provide representation for transcendental functions that are much more generally valid than the classical representation by, say, the power series.

Recently, the extension of continued fraction theory from real numbers to the matrix case has seen several developments and interesting applications (see [2], [5] and [6]). Since calculations involving matrix valued functions with matrix arguments are feasible with large computers, it will be an interesting attempt to develop such matrix theory. The real case is relatively well studied in the literature (see [7] and [8]). However, in contrast to the theoretical importance, one can find in mathematical literature only a few results on the continued fractions with matrix arguments ([10] and [12]).

The gamma function, defined by $\Gamma(a) = \int_0^{+\infty} t^{a-1}e^{-t}dt$ is the most important special function of classical analysis after the so-called elementary functions. It is an extension of the factorial $n!$ to real and complex arguments. It is related to the factorial by $\Gamma(n) = (n-1)!$.

The incomplete gamma function $\gamma(a, z)$ and the complementary incomplete gamma function $\Gamma(a, z)$ are generalizations of the gamma function. Note that the importance

of these functions lies in the fact that they can be expressed to calculate a probability distribution and its cumulative distribution function.
These generalizations satisfy the relation

$$\Gamma(a,z)+\gamma(a,z)=\Gamma(a),\ \mathcal{R}a>0,\ |argz|<\pi,$$

such that

$$\Gamma(a,z):=\int_z^{+\infty}t^{a-1}e^{-t}dt,\ a\in\mathbb{C},\ |argz|<\pi,$$

and

$$\gamma(a,z):=\int_0^z t^{a-1}e^{-t}dt,\ \mathcal{R}a>0,\ |argz|<\pi.$$

The present paper contains a continued fraction representation of the incomplete gamma function $\gamma(a,x)$ that is defined for real numbers $a>2$, $x>0$ in the real case and the matrix case. This function has numerous applications in statistics, probability theory, and other fields. The most important properties of this function are collected, for example, see chapter 6 of [1]. Much information on the incomplete gamma function with interesting historical comments and a detailed list of references can be found in [4].

The series expansion of the incomplete gamma function $\gamma(a,x)$ is given by, (see [2])

$$\gamma(a,x)=x^a\sum_{k=0}^{+\infty}\frac{(-1)^k}{(a+k)\,k!}x^k,\ a>0,x>0. \tag{1}$$

2. Preliminaries and notations

Throughout this paper, $\mathcal{M}_m(\mathbb{R})$ will represent an algebra of real matrices of sizes $m\times m$. Since the complex case can be stated similarly to the real case, then we limit our attention to the last case.

We now introduce some topological notions of continued fractions with matrix arguments. Let $A\in\mathcal{M}_m(\mathbb{R})$, we put

$$||A||=\sup_{x\neq0}\frac{||Ax||}{||x||}=\sup\{||Ax||,||x||=1\}.$$

This norm satisfies the inequality

$$||AB||\leq||A||\,||B||.$$

We now mention an important result of matrix functions. Let $A\in\mathcal{M}_m(\mathbb{R})$, A is said to be positive semidefinite (resp. positive definite) if A is symmetric and

$$\forall x\in\mathbb{R}^m,(Ax,x)\geq0\,(resp.\forall x\in\mathbb{R}^m,x\neq0\,(Ax,x)>0),$$

where $(.,.)$ denotes the standard scalar product of $\mathbb{R}^m$ defined by

$$\forall x=(x_1,...,x_m),y=(y_1,...,y_m)\in\mathcal{M}_m(\mathbb{R}),(x,y)=\sum_{i=1}^m x_iy_i.$$

We observe that positive semidefiniteness induces a partial ordering on the space of symmetric matrices: if A and B are two symmetric matrices, then we write $A \geq B$ if $B - A$ is positive semidefinite.

Henceforth, whenever we say that $A \in \mathcal{M}_m(\mathbb{R})$ is positive semidefinite (or positive definite), it will be assumed that A is symmetric. It is easy to see that if $A \leq B$ then $CAC \leq CBC$ for all symmetric matrix C.

For any $A, B \in \mathcal{M}_m(\mathbb{R})$ with B invertible, we write $A/B = B^{-1}A$, in particular, if $A = I$, the matrix identity, then $I/B = B^{-1}$. It is easy to verify that, if C and D two matrices with C invertible then in general,

$$\frac{A}{B} = \frac{CA}{CB} \neq \frac{AC}{BC},$$

$$C\frac{A}{B}D = \frac{AD}{BC^{-1}}.$$

Let (A_n) be a sequence of matrices in $\mathcal{M}_m(\mathbb{R})$. We say that (A_n) converges in $\mathcal{M}_m(\mathbb{R})$ if there exists a matrix $A \in \mathcal{M}_m(\mathbb{R})$ such that $\|A_n - A\|$ tends to 0 when n tends to $+\infty$. In this case we write, $A_n \to A$ or $\lim_{n\to+\infty} A_n = A$.

Definition 2.1 ([11])**.** *Let $(A_n)_{n\geq 0}$ and $(B_n)_{n\geq 1}$ be two sequences of matrices in $\mathcal{M}_m(\mathbb{R})$. We denote the continued fraction expansion by*

$$A_0 + \cfrac{B_1}{A_1 + \cfrac{B_2}{A_2 + \dots}} := \left[A_0; \frac{B_1}{A_1}, \frac{B_2}{A_2}, \dots\right].$$

Sometimes, we denote this continued fraction by $\left[A_0; \frac{B_n}{A_n}\right]_{n=1}^{+\infty}$ *or $K(B_n/A_n)$, where*

$$\left[A_0; \frac{B_i}{A_i}\right]_{i=1}^{n} = \left[A_0; \frac{B_1}{A_1}, \frac{B_2}{A_2}, \dots, \frac{B_n}{A_n}\right].$$

The fractions $\frac{B_n}{A_n}$ *and* $\frac{P_n}{Q_n} = \left[A_0; \frac{B_i}{A_i}\right]_{i=1}^{n}$ *are called, respectively, the n^{th} partial quotient and the n^{th} convergent (approximant) of the continued fraction $K(B_n/A_n)$. When $B_n = I$ for all $n \geq 1$, then $K(I/A_n)$ is called a simple continued fraction.*

The continued fraction $K(B_n/A_n)$ converges in $\mathcal{M}_m(\mathbb{R})$ if the sequence

$$(F_n) = \left(\frac{P_n}{Q_n}\right) = \left(Q_n^{-1}P_n\right)$$

converges in $\mathcal{M}_m(\mathbb{R})$ in the sense that there exists a matrix $F \in \mathcal{M}_m(\mathbb{R})$ such that

$$\lim_{n\to+\infty} \|F_n - F\| = 0.$$

In this case, we denote $F = \left[A_0; \frac{B_n}{A_n}\right]_{n=1}^{+\infty}$. *In the opposite case, we say that $K(B_n/A_n)$*

is divergent.

We note that the evaluation of n^{th} convergent according to Definition 1 is not practical because we have to repeatedly invert matrices. The following proposition gives an adequate method to calculate $K\left(B_n/A_n\right)$.

Proposition 2.1. *For the continued fraction $K\left(B_n/A_n\right)$, define*

$$\left\{\begin{array}{l} P_{-1}=I,\ P_0=A_0 \\ Q_{-1}=0,\ Q_0=I \end{array}\right. \quad \textit{and} \quad \left\{\begin{array}{l} P_n=A_n\ P_{n-1}+B_nP_{n-2} \\ Q_n=A_n\ Q_{n-1}+B_nQ_{n-2} \end{array}\right. \quad n\geq 1.$$

Proof. See [11]. □

Proposition 2.2. *For any two matrices C and D with C invertible, we have*

$$C\left[A_0;\frac{B_k}{A_k}\right]_{k=1}^{n} D=\left[CA_0D;\frac{B_1D}{A_1C^{-1}},\frac{B_2C^{-1}}{A_2},\frac{B_k}{A_k}\right]_{k=3}^{n}.$$

Proof. The proof of this proposition is elementary and we leave it to the reader. □

Definition 2.2. *Let $(A_n),(B_n)$ (C_n) and (D_n) be four sequences of matrices. We say that the continued fractions $K(B_n/A_n)$ and $K(D_n/C_n)$ are equivalent if we have $F_n=G_n$ for all $n\geq 1$, where F_n and G_n are the n^{th} convergents of $K(B_n/A_n)$ and $K(D_n/C_n)$ respectively.*

The following lemma characterizes the equivalence of continued fractions.

Lemma 2.3. *[6]. Let (r_n) be a non-zero sequence of real numbers. The continued fractions*

$$\left[a_0;\frac{r_1b_1}{r_1a_1},\frac{r_2r_1b_2}{r_2a_2},...,\frac{r_nr_{n-1}b_n}{r_na_n},...\right] \textit{ and } \left[a_0;\frac{b_1}{a_1},\frac{b_2}{a_2},...,\frac{b_n}{a_n},...\right]$$

are equivalent.

We also recall the following lemma. From the expansion of a function given by its Taylor series, we give the expansion in continued fractions of the series that was established by Euler.

Lemma 2.4. *Let f be a function with Taylor series expansion $f\left(x\right)=\sum_{n=0}^{+\infty}c_nx^n$ in $D\subset\mathbb{R}$. Then, the expansion in continued fraction of $f\left(x\right)$ is*

$$f\left(x\right)=\left[0;\frac{c_0}{1},\frac{-c_1x}{c_0+c_1x},\frac{-c_0c_2x}{c_1+c_2x},\frac{-c_1c_3x}{c_2+c_3x},...,\frac{-c_{n-2}c_nx}{c_{n-1}+c_nx},...\right].$$

Proof. See [7]. □

Remark 1. *Let (A_n) and (B_n) be two sequences of $\mathcal{M}_m\left(\mathbb{R}\right)$. Then we notice that we can write the first convergents of the continued fraction $K(B_n/A_n)$ by*

$$F_1=A_0+A_1^{-1}B_1=A_0+\left(B_1^{-1}A_1\right)^{-1},$$

$$F_2=A_0+\left(A_1+A_2^{-1}B_2\right)^{-1}B_1=A_0+\left(B_1^{-1}A_1+\left(B_2^{-1}A_2B_1\right)^{-1}\right)^{-1}.$$

If we put, $A_1^* = B_1^{-1}A_1$ *and* $A_2^* = B_2^{-1}A_2B_1$, *we have*

$$F_1 = A_0 + \frac{I}{A_1^*}, F_2 = A_0 + \frac{I}{A_1^* + \frac{I}{A_2^*}}.$$

Generally, we prove by a recurrence that if we put for all $k \geq 1$,

$$A_{2k}^* = (B_{2k}...B_2)^{-1} A_{2k}B_{2k-1}...B_1$$

and

$$A_{2k+1}^* = (B_{2k+1}...B_1)^{-1} A_{2k+1}B_{2k}...B_2,$$

then the continued fractions $A_0 + K(B_n/A_n)$ *and* $A_0 + K(I/A_n^*)$ *are equivalent.*

So, the convergence of one of these continued fractions implies the convergence of the other continued fraction.

We will use the following theorem to prove our main result.

Theorem 2.5. *Let* (A_n), (B_n) *be two sequences of* $\mathcal{M}_m(\mathbb{R})$.. *If*

$$\| (B_{2k-2}...B_2)^{-1}A_{2k-1}^{-1}B_{2k-1}...B_1 \| \leq \alpha$$

and

$$||(B_{2k-1}...B_1)^{-1}A_{2k}^{-1}B_{2k}...B_2|| \leq \beta$$

for all $k \geq 1$, *where* $0 < \alpha < 1, 0 < \beta < 1$ *and* $\alpha\beta \leq 1/4$, *then the continued fraction* $K(B_n/A_n)$ *converges in* $\mathcal{M}_m$.

Proof. See [11] pp. 126. □

We need to present the following Proposition.

Proposition 2.6 ([9])**.** *Let* $C \in \mathcal{M}_m(\mathbb{R})$. *such that* $||C|| < 1$, *then the matrix* $I - C$ *is invertible and we have*

$$||(I - C)^{-1}|| \leq \frac{1}{1 - ||C||}.$$

To end this section, we give the following Theorem.

Theorem 2.7 ([3])**.** *If the function* $f(x)$ *can be expanded in a power series in the interval* $| x - x_0 | < r$, *as*

$$f(x) = \sum_{p=0}^{+\infty} \alpha_p(x - x_0)^p,$$

then this expansion remains valid when the scalar argument x *is replaced by a matrix* A *whose characteristic values lie within the circle of convergence, and we have*

$$f(A) = \sum_{p=0}^{+\infty} \alpha_p (A - x_0I)^p.$$

3. Main results

3.1. **The real case.**

Theorem 3.1. *Let $x \in \mathbb{R}$ and $a > 1$. Then the continued fraction expansion of the incomplete gamma function is*

$$\gamma(a,x) = \left[0; \frac{x^a}{a}, \frac{a^2 x}{a+1-ax}, \frac{(n-2)(a+n-2)^2 x}{(a+n-1)(n-1)-(a+n-2)x}\right]_{n=3}^{+\infty}.$$

Proof. We use Lemma 2.4 for the function

$$g(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(a+n)\,n!} x^n,\ c_n = \frac{(-1)^n}{(a+n)\,n!}.$$

So, we have

$$\frac{c_0}{1} = \frac{\frac{1}{a}}{1},\quad \frac{-c_1 x}{c_0 + c_1 x} = \frac{\frac{x}{a+1}}{\frac{a+1-ax}{a(a+1)}}.$$

For $n \geq 3$, we get

$$\frac{-c_{n-3}c_{n-1}x}{c_{n-2}+c_{n-1}x} = \frac{\frac{-x}{(a+n-1)(a+n-3)(n-3)!(n-1)!}}{\frac{(-1)^{n-2}((a+n-1)(n-1)!-(a+n-2)(n-2)!x)}{(a+n-1)(a+n-2)(n-2)!(n-1)!}}.$$

Therefore, the continued fraction expansion of $\gamma(a,x) = x^a g(x)$ is

$$\begin{aligned}\gamma(a,x) &= \left[0; \frac{b_n}{a_n}\right]_{n=1}^{+\infty}\\ &= \left[0; \frac{\frac{x^a}{a}}{1}, \frac{\frac{x}{a+1}}{\frac{a+1-ax}{a(a+1)}}, \frac{\frac{-x}{(a+n-1)(a+n-3)(n-3)!(n-1)!}}{\frac{(-1)^{n-2}((a+n-1)(n-1)!-(a+n-2)(n-2)!x)}{(a+n-1)(a+n-2)(n-2)!(n-1)!}}\right]_{n=3}^{+\infty}.\end{aligned}$$

Let us define the sequence $(r_n)_{n\geq 1}$ by

$$\begin{cases} r_1 &= a, \\ r_n &= (-1)^n (a+n-1)(a+n-2)(n-1)! \ for\ n \geq 2. \end{cases}$$

Then, we have

$$\begin{cases} \frac{r_1 b_1}{r_1 a_1} &= \frac{x^a}{a}, \\ \frac{r_1 r_2 b_2}{r_2 a_2} &= \frac{a^2 x}{a+1-ax}, \\ \frac{r_{n-1} r_n b_n}{r_n a_n} &= \frac{(n-2)(a+n-2)^2 x}{(a+n-1)(n-1)-(a+n-2)x}, for\ n \geq 3. \end{cases}$$

By applying the result of Lemma 2.3 to the sequence $(r_n)_{n\geq 1}$, we obtain

$$\gamma(a,x) = \left[0; \frac{x^a}{a}, \frac{a^2 x}{a+1-ax}, \frac{(n-2)(a+n-2)^2 x}{(a+n-1)(n-1)-(a+n-2)x}\right]_{n=3}^{+\infty}.$$

and the proof is complete. □

3.2. **The matrix case.** According to Theorem 2.7, we have

Definition 3.1. *Let A be a matrix in $\mathcal{M}_m(\mathbb{R})$. Then we define the incomplete gamma function by the expression*

$$\gamma(a,A) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(a+n)\, n!} A^{n+a}.$$

Now, we treat the matrix case:

Theorem 3.2. *Let A be a matrix in $\mathcal{M}_m(\mathbb{R})$, such that $\|A\| = \alpha$, where $0 < \alpha < 1/2$, and $a > 1$. The continued fraction*

$$\left[0; \frac{A^a}{aI}, \frac{a^2 A}{(a+1)I-aA}, \frac{(n-2)(a+n-2)^2 A}{(a+n-1)(n-1)I-(a+n-2)A}\right]_{n=3}^{+\infty}.$$

converge in $\mathcal{M}_m(\mathbb{R})$. Furthermore, this continued fraction represents $\gamma(a,A)$. So

$$\gamma(a,A) = \left[0; \frac{A^a}{aI}, \frac{a^2 A}{(a+1)I-aA}, \frac{(n-2)(a+n-2)^2 A}{(a+n-1)(n-1)I-(a+n-2)A}\right]_{n=3}^{+\infty}.$$

Proof. We study the convergence of the continued fraction $K(B_k/A_k)$ with

$$\begin{cases} A_1 &= aI, \ \ A_2 = (a+1)I - aA, \\ B_1 &= A^a, \ \ B_2 = a^2 A, \end{cases}$$

and for $k \geq 3$, we have

$$\begin{cases} A_k &= (a+k-1)(k-1)I - (a+k-2)A, \\ B_k &= (k-2)(a+k-2)^2 A. \end{cases}$$

We check that the conditions of Theorem 2.5 are satisfied. We have

$$(B_{2k-2}B_{2k-4}...B_4)^{-1} = \frac{1}{(2k-4)(a+2k-4)^2(2k-6)(a+2k-6)^2...2(a+2)^2} A^{2-k},$$

$$B_{2k-3}B_{2k-5}...B_1 = (2k-3)(a+2k-3)^2(2k-5)(a+2k-5)^2...(a+1)^2 A^{k-2+a}$$

and

$$\begin{aligned} A_{2k-1}^{-1} &= ((a+2k-2)(2k-2)I - (a+2k-3)A)^{-1} \\ &= \frac{1}{(a+2k-2)(2k-2)}(I - \frac{a+2k-3}{(a+2k-2)(2k-2)}A)^{-1}. \end{aligned}$$

Since we have the matrices A^k and $(I-\frac{a+2k-3}{(a+2k-2)(2k-2)}A)^{-1}$ commute, then we obtain

$$\begin{aligned}\|\left(B_{2k-2}...B_4\right)^{-1}A_{2k-1}^{-1}B_{2k-3}...B_1\|=&\frac{1}{2}\times\frac{3}{4}\times...\times\frac{2k-5}{2k-4}\times(\frac{a+1}{a+2})^2\times(\frac{a+3}{a+4})^2\times...\times\\&(\frac{a+2k-5}{a+2k-4})^2\frac{1}{(a+2k-2)(2k-2)}\\&\|A^a(I-\frac{a+2k-3}{(a+2k-2)(2k-2)}A)^{-1}\|\\&\leq\|A^a(I-\frac{a+2k-3}{(a+2k-2)(2k-2)}A)^{-1}\|.\end{aligned}$$

By Proposition 2.6, for all sufficiently large k and the fact that $\|A\|< 1/2$, we obtain

$$\|(I-\frac{a+2k-3}{(a+2k-2)(2k-2)}A)^{-1}\|\leq\frac{1}{1-\frac{a+2k-3}{(a+2k-2)(2k-2)}\|A\|}<1$$

then, we get

$$\|(B_{2k-2}...B_4)^{-1}A_{2k-1}^{-1}B_{2k-3}...B_1\|\leq\|A\|^a<\left(\frac{1}{2}\right)^a<\frac{1}{2}.$$

In a similar way to the previous one, we show the second inequality of Theorem 2.5, we have

$$\begin{aligned}\|\left(B_{2k-1}...B_3\right)^{-1}A_{2k}^{-1}B_{2k-2}...B_2\|=&\frac{2}{3}\times\frac{4}{5}\times...\times\frac{2k-4}{2k-3}\times(\frac{a}{a+1})^2\times\\&(\frac{a+2}{a+3})^2\times...\times(\frac{a+2k-4}{a+2k-3})^2\frac{1}{(a+2k-1)(2k-1)}\\&\|A(I-\frac{a+2k-2}{(a+2k-1)(2k-1)}A)^{-1}\|\\&\leq\|A(I-\frac{a+2k-2}{(a+2k-1)(2k-1)}A)^{-1}\|\\&\leq\frac{1}{1-\frac{a+2k-2}{(a+2k-1)(2k-1)}\|A\|}\|A\|\\&\leq\|A\|\\&\leq\frac{1}{2}\end{aligned}$$

which completes the proof.

□

3.3. **Application of Theorem** 3.1. As an application of theorem 3.1, we give the following example. Let α,θ and x be three real numbers such that $\alpha>1,\theta>0,$ and

$x \geq 0$. A probability distribution

$$f(x;\alpha,\theta) = \frac{1}{\theta^{\alpha}\Gamma(\alpha)} e^{-x/\theta} x^{\alpha-1} \tag{2}$$

with parameters α and θ, is called a gamma distribution. The cumulative distribution function for the gamma distribution $f(x;\alpha,\theta)$ is

$$P(x;\alpha,\theta) = \frac{\gamma(\alpha, x/\theta)}{\Gamma(\alpha)}. \tag{3}$$

The function (3) is called the regularised gamma function (See [2]).

From equation (3) and (1), we obtain the series expansion

$$P(x;\alpha,\theta) = \frac{(x/\theta)^{\alpha}}{\Gamma(\alpha)} \sum_{k=0}^{+\infty} \frac{(-x/\theta)^{k}}{(\alpha+k)\,k!}. \tag{4}$$

Corollary 3.3. *Let $x \geq 0, \theta > 0$ and $\alpha > 1$. Then the continued fraction expansion of the regularised gamma function is*

$$P(x;\alpha,\theta) = \left[0; \frac{(x/\theta)^{\alpha}}{\alpha\Gamma(\alpha)}, \frac{\alpha^2\Gamma(\alpha)(x/\theta)}{\alpha+1-\alpha(x/\theta)}, \frac{(n-2)(\alpha+n-2)^2(x/\theta)}{(\alpha+n-1)(n-1)-(\alpha+n-2)(x/\theta)}\right]_{n=3}^{+\infty}.$$

4. Numerical applications

In the previous sections of this article, we have given the real and continued fraction matrix expansions of the incomplete gamma function. Now in this section, we will see some numerical data to illustrate these theoretical results.

Example 4.1.

In combination, the following sequence of tables describes the difference between the tabular value $\gamma(a, x)$ of the incomplete gamma function with its k^{th} partial sum and its k^{th} approximant where $k = 1; 5$ and 10 in the case $a = 4$.

x	$\gamma(a,x)-S_1(a,x)$	$\gamma(a,x)-F_1(a,x)$	$\gamma(a,x)-S_5(a,x)$	$\gamma(a,x)-F_5(a,x)$	$\gamma(a,x)-S_{10}(a,x)$	$(\gamma(a,x)-F_{10}(a,x)$
0.05	1.23e-6	6.12e-8	1.02e-6	-1.79e-15	5.61e-7	1.19e-26
0.1	1.91e-5	1.91e-6	1.61e-5	9.12e-13	9.20e-6	1.95e-22
0.2	2.83e-4	5.89e-5	2.39e-4	-4.60e-10	1.38e-4	3.17e-18
0.3	1.31e-3	.4.30e-4	1.09e-3	-1.74e-8	6.09e-4	9.18e-16
0.4	3.77e-3	1.74e-3	3.10e-3	-2.28e-7	1.55e-3	5.10e-14
0.5	8.30e-3	5.11e-3	6.63e-3	-1.68e-6	2.79e-3	1.15e-12

We note that the algorithm of the continued fraction converges very quickly than that of the partial sum for example for $x = 0.05$, we have 15 digits which coincide between the 5^th approximant and the tabular value of the incomplete gamma function for the method of continued fractions, while the number in the sum method of the same iteration was only 6.

Now, we illustrate the result obtained in the matrix case

Example 4.2. Let A be a matrix in $\mathcal{M}_3(\mathbb{R})$ such that

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{20} & \frac{1}{18} \\ \frac{1}{20} & \frac{1}{4} & \frac{1}{20} \\ \frac{1}{18} & \frac{1}{20} & \frac{1}{4} \end{pmatrix}.$$

The norm of A is 0.3625 and we therefore have $||A|| < 1/2$ which verifies the necessary condition of the theorem 3.2. We pose $a = 3$, we will calculate some values of $\gamma(a, A) - \gamma_n(a, A)$ by using the partial sum method based on definition 3.1 where $\gamma_n(a, A) = \sum_{k=0}^{n} \frac{(-1)^k}{(a+k)\, k!} A^{k+a}$ and some values of $\gamma(a, A) - F_n(a, A)$ by using the continued fraction expansion result that we got in Theorem 3.2.. We have the following results

- For $n = 1$, we have

$$F_1(a, A) - \gamma(a, A) = \begin{pmatrix} 1.46e-3 & 1.03e-3 & 1.17e-3 \\ 1.03e-3 & 1.35e-3 & .1.03e-3 \\ 1.03e-3 & 1.17e-3 & 1.46e-3 \end{pmatrix}$$

and we have

$$\gamma_1(a, A) - \gamma(a, A) = \begin{pmatrix} -6.65e-3 & -3.71e-3 & -4.34e-3 \\ -3.71e-3 & -6.36e-3 & -3.71e-3 \\ -4.34e-3 & -3.71e-3 & -6.65e-3 \end{pmatrix}$$

- For $n = 5$, we have

$$F_5(a, A) - \gamma(a, A) = \begin{pmatrix} 9.60e-8 & 8.63e-8 & 9.45e-8 \\ 8.63e-8 & 8.25e-8 & 8.63e-8 \\ 9.45e-8 & 8.63e-8 & 9.60e-8 \end{pmatrix}$$

and we have

$$\gamma_5(a, A) - \gamma(a, A) = \begin{pmatrix} -7.40e-3 & -4.14e-3 & -4.84e-3 \\ -4.14e-3 & -7.06e-3 & -4.14e-3 \\ -4.84e-3 & -4.14e-3 & -7.40e-3 \end{pmatrix}$$

- For $n = 10$, we have

$$F_{10}(a, A) - \gamma(a, A) = \begin{pmatrix} 1.17e-14 & 1.076e-14 & 1.16e-14 \\ 1.07e-14 & 9.93e-15 & 1.07e-14 \\ 1.16e-14 & 1.07e-14 & 1.17e-14 \end{pmatrix}$$

and we have

$$\gamma_{10}(a, A) - \gamma(a, A) = \begin{pmatrix} -8.791e-3 & -.4.93e-3 & -5.77e-3 \\ -4.93e-3 & -8.39e-3 & -4.93e-3 \\ -5.77e-3 & -4.93e-3 & -8.79e-3 \end{pmatrix}$$

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